WINNERS' GUIDE TO GMAT MATH - Part I

2012-13 Ed

BY

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CONTENTS

NOTE: The Content in **RED** is available in the unabridged copy of **'Winners' Guide to GMAT Math – Part 1 2012-13 Ed'** available at:

http://winningprep.com/WinnersGuideToGMATMath1.html

INTRODUCTION

GMAT Math is easy! Well that is what most students with a technical background think. But if you are serious about a 90 percentile plus score in math, you simply cannot afford to be complacent.

On the other hand, there are students who shudder at the very thought of facing a math problem. Well, neither of these approaches is conducive to a high math score.

What is required is a thorough study of the **fundamentals**, a basic grasp of the **concepts**, and developing an ability to **apply** these concepts to the gmat type problems.

Then comes the ability to solve a gmat problem in **multiple** ways, the ability to use **shortcuts** when stumped, and the ability to **guess intelligently.**

Whether you are a novice or a math expert, you *do* need to brush up /build up your fundamentals, and then go on to the tougher problems.

And this is exactly what this book does.

It helps you to develop a solid **understanding of the underlying concepts**, builds upon this understanding by providing various **different types of examples**, exposes you to **alternative ways** of looking at a particular problem, and finally shows you **how to use shortcuts**.

THE BASICS

Before we launch straight into the business of learning math, it's a good idea to revisit the basics, the building blocks of math. Without these basics, any further study or calculation is impossible, and it's impossible to correctly solve equations or problems without these building blocks.

SYMBOLS

The first step in solving math problems is to understand the symbols that are used in equations. For this reason, we've listed some of the common symbols you will encounter in your quest to master mathematics.

Symbol	Denotes
=	Equal to
+	Plus
-	Minus
÷	Divide
≠	Not equal to
≤	Smaller than or equal to
≥	Greater than or equal to
<	Greater than
>	Smaller than
→	Strives to
∞	Infinity
E	2.71828183…

There will of course be other symbols you will encounter in specific sections of your math studies, however, knowing these basics will make your studies a lot easier, and your understanding, and ability to solve problems, much greater.

VARIABLES

Variables represent the unknown. For example, in the problem $|3 + x| = 5$, the variable, x, would represent 2, or it may represent -8.

(**Note:** $|x|$ represents the absolute value of x)

So, when we calculate or solve an algebraic problem, all we are really doing is applying mathematical rules to discover what the unknown or variable is.

COEFFICIENTS

If any number is multiplied by a variable, the number is known as the coefficient. For example:

$3a = 3 \times a$

In this instance, 3 is the coefficient of a.

Remember, if you see a variable that has no coefficient, the coefficient is 1.

SIMILAR TERMS

Simple algebraic equations and problems use similar terms. This is where the variable of the terms are the same. For example:

a, 2a, $\frac{1}{2}$a while the coefficient differs, the variable of all terms is the same

On the other hand, an example of non-similar terms would be:

2a, 3b, 2x.

ADDITION AND SUBTRACTION OF ALGEBRAIC TERMS

When you add or subtract algebraic terms, only those with similar variables can be subtracted. For example:

$2a – 4a = -2a$

SQUARES, SQUARE- ROOTS, SURDS & INDICES

SQUARE OF A NUMBER

The square of a number is found out by multiplying the number by itself e.g.

Square of 56 = 56 × 56 = 56^2 = 3136.

Square of -56 = -56 × -56 = $(-56)^2$ = 3136.

Thus, the square of any nonzero number is always positive.

Perfect square: The Square of any natural number is called a perfect square.

Example: $1^2 = 1$; $2^2 = 4$; $3^2 = 9$; $4^2 = 16$ etc.

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121 … are all perfect squares.

PROPERTIES OF SQUARE NUMBERS

i) The square of a positive number greater than 1 is greater than the number itself. e.g., $3^2 = 9$ which is greater than 3.
ii) The square of a positive number smaller than 1 is smaller than the number itself. e.g., $(0.3)^2$ = 0 09 which is smaller than 0.3.
iii) The square of a negative number is always greater than the number as the square is always positive.
iv) A square cannot end in an odd number of zeros.
v) The squares of the first ten natural numbers end 0, 1, 4, 5, 6 and 9. Hence, a number ending with 2, 3, 7 or 8 can never be a perfect square.
vi) The square of an odd number is odd.
vii) The square of an even number is even.
viii) Every square number is a multiple of 3, or exceeds a multiple of 3 by one.
ix) Every square number is a multiple of 4 or exceeds a multiple of 4 by one.
x) If a square number ends in 1 or 9, the preceding digit is not odd.
xi) If a square number ends in 6, the preceding digit is odd.
xii) If a square number ends is 5, the preceding digit is 2.

SQUARE ROOT

The square root of a given number is a number whose square is equal to the given number. e.g., The square root of 9 is 3.

Square root is denoted by the symbol√. There are two possibilities for the square root of a positive number, one positive and one negative. **Only the positive one is called the square root.**

Thus, $\sqrt{49}$ =7 even through (-7) × (-7) = 49

As the square of any nonzero number is positive, the square root of a negative number is not defined as a real number.

Thus, $\sqrt{-4}$ is not a real number.

The number of digits in the square root of a number with an even number of digits is half of the number of digits in the number.

Example: $\sqrt{1600}$ = 400, there are four digits in 1600 and two digits in 40.

The number of digits in the square root of a number with odd number of digits is half of the number obtained by adding 1 to the number of digits in the given number.

Example: $\sqrt{900}$ = 30, there are three digits in 900 and two digits in 30.

Methods of finding square root of a given number

BY PRIME FACTORIZATION

Factorize the given number. If the number is a perfect square, there will be an exact number of prime factors. The product of one factor from each pair gives the square root of the number.

Example: i) $\sqrt{44100} = \sqrt{7^2 \times 3^2 \times 2^2 \times 5^2} = \sqrt{21^2 \times 10^2} = 21 \times 10 = 210$

ii) $\sqrt{216} = \sqrt{2 \times 2 \times 2 \times 3 \times 3 \times 3} = 6\sqrt{6}$

BY DIVISION

Let us find the square root of 46656.

First, divide the number to be square-rooted into pairs of digits, starting at the decimal point. That is, no digit pair should straddle a decimal point. (For example, split 1225 into "12 25" rather than "1 22 5"; 6.5536 into "6. 55 36" rather than"6.5 53 6".).

Then you can put some lines over each digit pair, and a bar to the left, somewhat as in long division.

```
   +--- ---- ----
   | 4  66  56
```

Find the largest number whose square is less than or equal to the leading digit pair. In this case, the leading digit pair is 4; the largest number whose square is less than or equal to 4 is 2.

Put that number on the left side, and above the first digit pair.

```
     2
   +--- ---- ----
 2 | 4  66  56
```

Now square that number, and subtract from the leading digit pair.

```
     2
   +--- ---- ----
 2 | 4  66  56
   |-4
   +----
     0
```

Extend the left bracket; multiply the last (and only) digit of the left-hand number by 2, put it to the left of the difference you just calculated, and leave an empty decimal place next to it.

```
     2
   +--- ---- ----
 2 | 4  66  56
   |-4
   +----
4_ | 0
```

Then bring down the next digit pair and put it to the right of the difference.

```
     2
   +--- ---- ----
 2 | 4  66  56
   |-4
   +----
4_ | 0  66
```

Find the largest number to put in this blank decimal place such that that number, times the number already there plus the decimal place, will be less than the current difference. For example, see if 1 * 41 is ≤ 66, then 2*42 ≤ 66, etc. In this case it's a 1. Put this number in the blank you left, and in the next decimal place on the result row on the top.

```
    2   1
   +--- ---- ----
 2 |4   66   56
   |-4
   +----
41 |0   66
```

Now subtract the product you just found.

```
    2   1
   +--- ---- ----
 2 |4   66   56
   |-4
   +----
41| 0   66
   |-   41
   +--------
        25
```

Now, repeat as before: Take the number in the left column (here, 41) and double its last digit (giving you 42). Copy this below in the left column, and leave a blank space next to it. (Double the last digit with carry: for example, if you had not 41 but 49, which is 40+9, you should copy down 40+18 which is 58.) Also, bring down the next digit pair on the right.

```
    2   1
   +--- ---- ----
 2 |4   66   56
   |-4
   +----
41 |0   66
   |-   41
   +--------
42_       25   56
```

Now, find the largest digit (call it #) such that 42# * # ≤ 2556. Here, it turns out that 426 * 6 = 2556 exactly.

```
     2   1   6
    +--- ---- ----
 2  | 4   66   56
    |-4
    +----
41| 0   66
    |-   41
    +--------
426|    25   56
    |-   25   56
    +-------------
             0
```

When the difference is zero, you have an exact square root and you're done. Otherwise, you can keep finding more decimal places for as long as you want.

Hence, the square root of 46656 is 216.

Here is another example, with less annotation. Let us find the square root of 53.

```
          7 .  2  8  0  1 ...
        +----------------------
7       | 53 . 00 00 00 00 00
        | 49
        +----------------------
142 |  4   00
        |  2   84
        +----------------------
1448 |1   16 00
        |  1   15 84
        +----------------------
14560|        16 00
        |           0
        +----------------------
145601|        16 00 00
        |           14 56 01
        +----------------------
        |           1 43 99 00
                      ...
```

SQUARE ROOT OF FRACTION

The square root of a fraction, if the denominator is a perfect square, is found by taking the square root of the numerator and denominator separately .e.g. $\sqrt{\frac{3}{16}} = \frac{\sqrt{3}}{4} = \frac{1}{4}\sqrt{3}$

In case of a mixed fraction, it must first be expressed as an improper fraction.

In case of a fraction whose denominator is not a perfect square, we may either convert the fraction into a decimal as a first step, or multiply the numerator and denominator by a number that will make the denominator a perfect square.

CUBE OF A NUMBER

The cube of a number is found out by multiplying the number with the square of that number. **Example:** $16^3 = 16^2 \text{ x } 16 = 4096$

The cube of a positive number is positive and the cube of a negative number is negative,

Example: $2^3 = 8$ and $(-2)^3 = -8$

CUBE ROOT

The cube root of a given number is the number whose cube is equal to the given number. Cube root of negative numbers exists. The cube root of a positive number is positive and the cube root of a negative number is negative. Cube root is denoted by the symbol $\sqrt[3]{}$

Example: $\sqrt[3]{8} = 2$ and $\sqrt[3]{-8} = -2$

To find the cube root by prime factorization:

Factorize the given number. Form groups of three of the same factors. The product of one factor from each group gives the cube root of the number.

Example:

i) $\sqrt[3]{287496} = \sqrt[3]{3 \times 3 \times 3 \times 2 \times 2 \times 2 \times 11 \times 11 \times 11} = \sqrt[3]{3^3 \times 2^3 \times 11^3} = 3 \times 2 \times 11 = 66.$

ii) $\sqrt[3]{5400} = \sqrt[3]{2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5} = 2 \times 3\sqrt[3]{5 \times 5} = 6\sqrt[3]{25}.$

INDICES

If a is any number and m is any positive integer then, a^m means a x a … m times i.e., the product of 'a' m times 'a' is called the base and m is called the power or exponent.

$\sqrt[n]{a}$ or $a^{\frac{1}{n}}$ denotes the nth root of a.

Example: $49^{\frac{1}{2}} = \sqrt{49} = 7$

$\sqrt[n]{a^m}$ or $a^{\frac{m}{n}}$ where $n \neq 0$, denotes the m^{th} power of n^{th} root of a, or n^{th} root of m^{th} power of a.

Example: $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = \left(\sqrt{4}\right)^3 = 2^3 = 8 \text{ } or \text{ } 4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = \sqrt{64} = 8$

The following are some of the important results for Indices:

i) **Multiplying numbers with same base:** To multiply numbers with same base, simply add the exponents of the base.

$$a^m \times a^n = a^{m+n}$$

$$a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\cdots}$$

ii) **Dividing numbers with same base:** To divide numbers with same base, simply subtract the exponent of the denominator from the exponent of the numerator.

$$a^m \div a^n = a^{m-n}$$
$$7^4 \div 7^3 = 7^{4-3} = 7$$

iii) **Raising a power to a power:** To raise a power to power, simply multiply the exponents.

$$(a^m)^n = a^{mn}$$
$$(3^2)^3 = 3^{2\times 3} = 3^6$$

iv) **Negative power of a base:** The n^{th} negative power of a number is equal to the reciprocal of the same number to the positive power n.

$$a^{-n} = \frac{1}{a^n}$$

$$27^{-3} = \frac{1}{27^3}$$

v) **Any nonzero number raised to the power 0 is equal to 1.**

$$\boldsymbol{a^0 = 1}$$

$$\boldsymbol{2^0 = 1}$$

vi) **Parenthesis:** When the only operation within parenthesis is multiplication and division, the exponent outside the parenthesis should be distributed to all the numbers within.

$(a \text{ x } b)^m = a^m \text{ x } b^m$ e.g., $(2 \text{ x } 3)^4 = 2^4 \text{ x } 3^4$;

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}\ e.g.\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$$

PRACTICE PROBLEMS

1. Find the squares of the following two numbers

$$3\frac{1}{2}, 6\frac{1}{2}$$

Solution:

The Square will be found out by multiplying the integral portion by next higher integer and adding $\frac{1}{4}$

$$\left(3\frac{1}{2}\right)^2 = 3 \times 4 + \frac{1}{4} = 12\ \frac{1}{4}$$

Similarly,

$$\left(6\frac{1}{2}\right)^2 = 6 \times 7 + \frac{1}{4} = 42\ \frac{1}{4}$$

2. Solve for x, if

$2^{2x+3} = 2^{2x} + 7168$

Solution :

$2^{2x+3} – 2^{2x} = 7168$

$2^{2x}(2^3 – 1) = 7168$

$$2^{2x}(7) = 7168;\ 2^{2x} = \frac{7168}{7};\ 2^{2x} = 1024$$

$$2^{2x} = 2^{10}; 2x = 10; x = 5$$

3. Simplify $\sqrt{\frac{36x^4y^2}{49y^4}}$

Solution:

$$\sqrt{\frac{36x^4y^2}{49y^4}} = \frac{6x^2}{7y}$$

4. Find the value of

$\sqrt{147} + \sqrt[3]{625} - \sqrt[3]{135} + \sqrt{27} + \sqrt[3]{40}$

Solution:

$\sqrt{49 \times 3} + \sqrt[3]{125 \times 5} - \sqrt[3]{27 \times 5} + \sqrt{9 \times 3} + \sqrt[3]{8 \times 5}$

$7\sqrt{3} + 5\sqrt[3]{5} - 3\sqrt[3]{5} + 3\sqrt{3} + 2\sqrt[3]{5} = 10\sqrt{3} + 4\sqrt[3]{5} = 2(5\sqrt{3} + 2\sqrt[3]{5})$

5. Which is greater $\sqrt[3]{30}$ or $\sqrt[4]{50}$?

Solution:

$\sqrt[3]{30} = \sqrt[12]{(30)^4} = \sqrt[12]{810000}$

$\sqrt[4]{50} = \sqrt[12]{(50)^3} = \sqrt[12]{125000}$

(Since LCM of 3 and 4 is 12, we convert both the surds to order 12)

Since 125000 < 810000, $\sqrt[4]{50} < \sqrt[3]{30}$

6. a and b are positive integers such that a > b. a is even and b is odd. m is a negative integer. Which of the following is/are true:

i) $\frac{m^a}{m^b}$ is positive   ii) $(m^a)^b$ is positive   iii) $m^a \times m^b$ is positive.

Solution:

i) $\frac{m^a}{m^b} = m^{a-b}$

a – b is an odd positive integer as a is even, b is odd and a > b.

i) m^{a-b} will be negative. Hence, i) is false.

ii) $(m^a)^b$ = $m^{a \text{ x } b}$

a x b is even. Hence, $m^{a \text{ x } b}$ will be positive. Hence, ii) is true.

iii) m^a x m^b = $m^{a + b}$

a + b is odd. Hence, $m^{a + b}$ will be negative. Hence, iii) is false.

7. If m^{12} is odd, then which of the following must be false?

(i) m^{13} is odd   (ii) m^2 is even   (iii) m^2 is odd

(iv) m is odd   (v) m^5 is odd

Solution:

Since m^{12} is odd, m is odd, as $odd^{even} = odd$

Therefore, m^2 is even is false.

Rest all is true. $\therefore$ (ii) is false.

8. If p and q are positive integers and $\sqrt{pq} = 25$, what could be the maximum value of p + q?

Solution:

$\sqrt{pq} = 25 \;\therefore pq = (25)^2 = 625$

Now, various factors of 625 = 5 x 125; 25 x 25; 1 x 625

Therefore, Maximum value of p + q could be 1 + 625 = 626

9. If we subtract 2 from a number, it becomes a perfect square. Which of the following **cannot** be the unit's digit of original number?

i) 2 ii) 5 iii) 9

iv) 4 v) none of these.

Solution:

A square number can never end in **2, 3, 7** or **8**

Therefore, i) 2 as unit's digit is possible.

ii) 5 as unit's digit is not possible.

iii) 9 as unit's digit is not possible.

iv) 4 as unit's digit is not possible

as 5 – 2 = **3**; 9 – 2 = **7**; 4 – 2 = **2**

Therefore, ii), iii), iv) are not possible.

10. Find the largest cube that is factor of both 576 and 2304.

Solution :

$576 = 24^2$

$= 24 \times 24 = 4 \times 6 \times 4 \times 6$

$= 4 \times 2 \times 3 \times 4 \times 2 \times 3 = 4 \times 4 \times 4 \times 3 \times 3$

$= 4^3 \times 9$

$2304 = 482^{2}$

$= 16^2 \times 3^2 = 4 \times 4 \times 4 \times 4 \times 9$

$= 4^3 \times 4 \times 9$

The largest cube that is a factor of both 576 and 2304 is $4^3 = 64$

11. A! is defined as a × (a – 1) × (a – 2) × …. × 1

What is the maximum power of 5 that can be extracted from 25!?

Solution :

25! = 25 × ……. × 20 × ……× 15 × ….. × 10 × …… × 5 × …

= 5 x 5 × … × 4 × 5 × … ×3 × 5 ×. .. × 2 × 5 × … × 5 × …

Therefore, 6 is the maximum power of 5 that can be extracted from 25!

12. $(\frac{1}{2})^{-3}\ (\frac{1}{4})^{-2}\ (\frac{1}{16})^{-1} =$

i) $(\frac{1}{2})^{-18}$ ii) $(\frac{1}{2})^{-11}$ iii) $(\frac{1}{2})^{-6}$

iv) $(\frac{1}{8})^{-11}$ v) $(\frac{1}{8})^{-6}$

Solution:

Given expression = $(\frac{1}{2})^{-3}\ (\frac{1}{4})^{-2}\ (\frac{1}{16})^{-1}$

Let us write the given expression with common denominator

$\left(\frac{1}{2}\right)^{-3}\left(\frac{1}{4}\right)^{-2}\left(\frac{1}{16}\right)^{-1} = \left(\frac{1}{2}\right)^{-3}\left(\frac{1}{2}\right)^{-4}\left(\frac{1}{2}\right)^{-4}$

$(\frac{1}{2})^{-3-4-4} = (\frac{1}{2})^{-11}$, Therefore ii)… is the answer.

13. Which of the following is the best approximation of $\sqrt{\frac{2.697}{0.032}} - 9.03$?

i) 0.13 ii) 2.9 ii) 3.7
iv) 5 v) 3

Solution:

$\sqrt{\frac{2.697}{0.032}} - 9.03 = \sqrt{\frac{2697}{32}} - 9.03 = \sqrt{84} - 9 = 9.16 - 9.03 = 0.13$

If you would rather not calculate the square root of 84, you know that it lies somewhere between 9 and 10, so the answer should be between -0.3 and 1. There is only one choice that meets these conditions.

AVERAGES

The GMAT uses the concept of averages to set convoluted problems that will not be found in most textbooks. While studying this chapter, make sure that you understand the techniques of faster computation. These techniques are essentially based on certain concepts and understanding them will, in turn, help you get a better grip on these concepts.

The average of a number is a measure of the central tendency of a set of numbers.

The average of a number is a measure of the central tendency of a set of numbers. In other words, it is an estimate of where the center point of a set of numbers lies.

The average or arithmetic mean of N numbers is the sum of the N numbers divided by N.

$$Average = \frac{Sum\ of\ terms}{Number\ of\ terms}$$

Example: find the average of 3, 4, 6, 2, 7, and 8.

$$Average = \frac{3+4+6+2+7+8}{6} = \frac{30}{6} = 5$$

MODE

The number that appears most frequently among N numbers is called the mode of the N numbers.

The number that appears most frequently among N numbers is called the mode of the N numbers.

Example: 1, 2, 2, 4, 5, 8, 2, 5, 6, 1, 6, 2, 4, 8

Here the mode is '2' as the number 2 appears most frequently.

If more than one number appears with the same maximum frequency the set does not have a unique mode.

MEDIAN

When N numbers are arranged in ascending or descending order the middle number is called the median.

If N is even there is no single middle number, the median is found by adding the two middle numbers and dividing the result by 2.

Example: find the median of 3, 4, 6, 2, 7, and 8.

Arranging the numbers in order we get 2, 3, 4, 6, 7 and 8. The middle two numbers are 4 & 6. The median is $\frac{4+6}{2} = 5$

ARITHMETIC MEAN

The **average or arithmetic mean** of N numbers is the sum of the N numbers divided by N.

$$Average = \frac{Sum\ of\ terms}{Number\ of\ terms}$$

The arithmetic mean of N numbers is the sum of the N numbers divided by N.

Example: find the mean of 3, 4, 6, 2, 7, and 8.

$$Average = \frac{3+4+6+2+7+8}{6} = \frac{30}{6} = 5$$

WEIGHTED ARITHMETIC MEAN

Sometimes the numbers among the terms to be averaged occur more than once. These numbers must be given the appropriate weight.

For example, If Kevin scored 90 in four subjects and 75 in two subjects then the average of Kevin's marks is not the sum of 90 and 75 divided by 2, but the average of 90, 90, 90, 90, 75 and 75.

To find the weighted average,

i) List the quantities and their respective weights.

   In the above example the quantities are 90 and 75 and their weights are 4 and 2.

ii) Multiply the value of each quantity by its respective weight.

   i.e., 90×4 and 75×2

iii) Add up the products.

   i.e., 360 + 150 = 510

iv) Add up the weights

i.e., 4 + 2 = 6

v) Divide the sum of the products by the sum of the weights.

Weighted average = $\frac{510}{6}$ = 85

GEOMETRIC MEAN

The geometric mean of N numbers is the N^{th} root of their products.

Example: Geometric mean of 3, 4 and 18 is $\sqrt[3]{3\times4\times18}$ = 6

Note: For N given numbers their arithmetic mean is always greater than their geometric mean.

> The geometric mean of N numbers is the Nth root of their products.

PROPERTIES OF AVERAGE (AM)

The properties of averages can be elucidated by the following examples:

Example: The average of 4 numbers 12, 13, 17 and 18 is:

Solution:

Required average = (12 + 13 + 17 + 18)/4 = 60/4 = 15

This means that if each of the 4 numbers of the set were replaced by 15 each, there would be no change in the total.

This is an important way to look at averages. This can be visualized as

12 → +3 → 15

13 → +2 → 15

17 → – 2 → 15

18 → – 3 → 15

60 → +0 → 60

i) In the above Example, visualize addition of a fifth number, which increases the average by 1.

15 + 1 = 16

15 + I = 16

15 + 1 = 16

15 + 1 = 16

The +1 appearing 4 times is due to the fifth number, which is able to maintain the average of 16 first and then 'give one' to each of the first 4.

Hence, the fifth number in this case is 16 + 4 = 20

ii) The average always lies above the lowest number of the set and below the highest number of the set.

iii) The net deficit due to the numbers below the average always equals the net surplus due to the numbers above the average.

iv) Ages and averages: If the average age of a group of persons is x years today then after n years their average age will be (x + n).

Also, n years ago their average age would have been (x – n). This happens due to the fact that for a group of people, 1 year is added to *each* person's age every year.

AVERAGE SPEED

Example: A man travels at 60 kmph on the journey from A to B and returns at 100 kmph. Find his average speed for the journey.

Solution:

Average speed = (total distance) / (total time)

If we assume distance between 2 points to be d

Then

Average speed = 2d/ [d/60 + (d/100)] = (2 x 60 x 100)/(60 + 100) = (2 x 60 x 100)/160 = 75

Average speed = $(2S_1.S_2) / (S_1 + S_2)$

S1 and S2 are speeds of going and coming back respectively.

SHORT CUT TO CALCULATE AVERAGE SPEED

The average speed will always come out by the following process:

The ratio of speeds is 60: 100 = 3:5 (say $r_1 : r_2$)

Then, divide the difference of speeds (40 in this case) by $r_1 + r_2$ (3 + 5 = 8, in this case) to get one part. (40/8 = 5, in this case).

The required answer will be three parts away (i.e. r_1 parts away) from the lower speed.

Check out how this works with the following speeds:

$S_1 = 20$ and $S_2 = 40$

Step 1: Ratio of speeds = 20 : 40 = 1: 2

Step 2: Divide difference of 20 into 3 parts ($r_1 + r_2$) → = 20/3 = 6.66

Required average speed = 20 + 1 x 6.66 = 26.66

If a body travels a particular distance while changing its speeds at various intervals of time then, its average speed is calculated by dividing the total distance covered by the total time taken.

i.e., if d_1, d_2, d_3 ..., d_n are the various distances covered in times t_1, t_2, t_3, ..., t_n with speeds s_1, s_2, s_3, ..., s_n, then the average speed for the journey is calculated by:

Average Speed =

$$\frac{Total\ distance\ traveled}{Total\ time\ taken} = \frac{d_1 + d_2 + d_3 + d_4 + d_5 + d_6 \ldots d_n}{t_1 + t_2 + t_3 + t_4 + \cdots t_n}$$

$$= \frac{d_1 + d_2 + d_3 + d_4 + d_5 + d_6 \ldots d_n}{\frac{d_1}{s_1} + \frac{d_2}{s_2} + \frac{d_3}{s_3} + \frac{d_4}{s_4} \ldots + \frac{d_n}{s_n}}$$

If a certain distance 'd' between two points A and B is covered at a speed of 's_1' miles/hr and same distance from B to A is covered at a speed of 's_2' miles/hr, then

Average speed = $\left(\frac{2s_1s_2}{s_1+s_2}\right)$

Some important points

i) If the distances d_1 and d_2 are covered in times t and t at the speeds of s_1 and s_2 respectively, then total time taken, T = $t_1 + t_2 = \frac{d_1}{s_1} + \frac{d_2}{s_2}$

ii) The time taken to travel a particular distance varies inversely with the speed i.e., the total time is reduced if the speed is increased, and it is increased if the speed is reduced. Alternatively, If a man changes his speed in the ratio m : n while travelling through a particular distance, then the ratio of time taken becomes n : m.

PRACTICE PROBLEMS

1. In a company of 30 employees the average salary is P. If the management decides to raise each employee salary by$10. What will the new average be?

i) $P + 3$ ii) $P + 10$ iii) $P + 30$

iv) $P + 300$ v) $10p$

Ans: ii

Solution:

This question can be solved without any calculations at all. Since the salary of ALL employees goes up by $10, that implies that the average salary **goes up by** $10, hence the new average = P + 10

Here is the formal way to solve the problem:

Let the total salay of all employees put together be x

Average salary of the employees=$\frac{x}{30} = P$

If the management decides to raise each employee salary, total salary of all employees:

$= 300 + \mathrm{x}$

Average=$\frac{300+\mathrm{x}}{30} = 10 + \frac{\mathrm{x}}{30} = 10 + \mathrm{P}$

2. One hour after Yolanda started walking from X to Y, a distance of 45 miles. Bob started walking along the same road from Y to X. If Yolanda's walking rate was 3 miles per hour and Bob's was 4 miles per hour, how many miles had Bob walked, when they met?

i) 24 ii) 23 iii) 22

iv) 21 v) 19.5

Ans: i

Solution:

Distance from X to Y = 45 $miles$

Yolanda's walking rate = 3 mph

Bob's Walking Rate = 4 mph

Relative walking rate of Yolanda and Bob =

$4 + 3 = 7\ mph\ (Walking\ in\ opposite\ direction)$

Distance covered by Yolanda in 1 hr = 4 $miles$

Remaining distance when Bob started walking = $45 - 3 = 42\ miles$

Time taken by Bob to meet Yolanda = $\frac{42}{7} = 6$ hours

Distance Bob will cover in 6 hours at his rate = $6\ \times 4 = 24\ miles$

3. In the first hour of two hour trip Paul travelled d kilometers and in the second hour he travelled one-half that distance. What is the average rate at which Paul travelled during the trip in $Kmph$?

i) d ii) $\frac{1}{3}d$ iii) $\frac{1}{2}d$

iv) $\frac{3}{4}d$ v) $\frac{3}{2}d$

Ans: iv

Solution:

Paul travelled d kilometers in 1 hour with speed s

$$1 = \frac{d}{s}$$

Paul travelled $\frac{d}{2}$ kilometers in 1 hour with speed s

$$1 = \frac{d}{2s}$$

Average speed=$s1 + s2$

$$1 + 1 = \frac{d}{s} + \frac{d}{2s} = \frac{2d + d}{2s}$$

$$2 = \frac{3d}{2s}$$

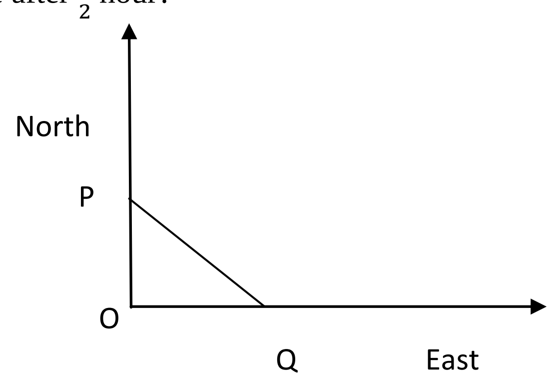
$$s = \frac{3d}{4}$$

Note: You can look at the problem as requiring you to find the average of d (speed during the first part) and d/2, i.e. speed during the second part.

4. Two walkers leave at the same time from the intersection of a point. First one started walking towards north at a constant rate of 8 miles per hour while the second one started walking towards east at a constant rate 4 miles per hour faster than the first walker's rate. How far apart, to the nearest mile, will they be after $\frac{1}{2}$ hour?

i) 6 ii) 7 iii) 8

iv) 12 v) 14



Ans: ii

Solution:

Refer the diagram.

Let the two walkers started from O .

Speed of the first walker is $= 8mph$

Speed of the second walker$= 12mph$ (4 miles per hour faster than the first walker's rate)

So after half an hour first walker will complete walking 4 miles towards north and reaches point P and second walker will complete walking 6 miles towards east and reaches Q.

From the Pythagorean Theorem $(PQ)^2 = (OQ)^2 + (OP)^2$

$(PQ)^2 = 6^2+4^2$

$(PQ)^2 = 36 + 16 = 52$

$PQ = \sqrt{52} = 7.2 = 7$ (Nearest Value)

5. On a recent trip Lynn drove her car 290 miles rounded to the nearest 10 miles and used 12 gallons of gasoline rounded to the nearest gallon. The actual number of miles per gallon that Lynn's car got on this trip must have been between.

i) $\frac{290}{12.5} and \frac{290}{11.5}$ ii) $\frac{295}{12} and \frac{285}{11.5}$ iii) $\frac{285}{12} and \frac{295}{12}$

iv) $\frac{285}{12.5} and \frac{295}{11.5}$ v) $\frac{295}{12.5} and \frac{285}{11.5}$

Ans: iv

Solution:

The lowest number of miles per gallon can be calculated using the lowest possible miles and the highest amount of gasoline. Also, the highest number of miles per gallon can be calculated using the highest possible miles and the lowest amount of gasoline.

Since, the miles are rounded to the nearest 10 miles; the number of miles is between 285 and 295.

Since the gasoline is rounded to the nearest gallon, the number of gallons is between 11.5 and 12.5.

So, the lowest number of miles per gallon is $\frac{285}{12.5}$, and the lowest number of miles per gallon is $\frac{295}{11.5}$

6. The average of a batsman after 25 innings was 56 runs per innings. If after the 26th inning his average increased by 2 runs, then how many runs did he score in the 26th inning?

Solution:

Runs in 26th inning = Runs total after 26 innings – Runs total after 25 innings

= 26 × 58 – 25 × 56

For quick calculation, rewrite the equation as:

(56 + 2) × 26 – 56 × 25

=2 × 26 + (56 × 26 – 56 × 25)

= 52 + 56 = **108**

Alternatively:

Since the average increases by 2 runs per innings it is equivalent to 2 runs being added to each score in the first 25 innings. Now, since these runs can only be added by the runs scored in the 26th inning, the score in the 26th inning must be 25 × 2 = 50 runs higher than the average after 26 innings (i.e. new average = 58).

Hence, runs scored in 26th inning = New Average + Old innings × Change in average

= 58 + 25 × 2 = **108**

7. The average age of a class of 30 students and a teacher reduces by 0.5 years if we exclude the teacher. If the initial average is 14 years, find the age of the class teacher.

Solution:

Age of teacher = Total age of (students + teacher) – Total age of students

= 31 × 14 – 30 × 13.5

= 434 – 405

= 29 years

Alternatively:

The teacher after fulfilling the average of 14 (for the group to which he belonged) is also able to give 0.5 years to the age of each of the 30 students. Hence, he has 30 × 0.5 → 15 years to give over and above maintaining his own average age of 14 years

Age of teacher = 14 + 30 × 0.5 = 29 years

(Note: This problem should be viewed as change of average from 13.5 to 14 when teacher is included.)